e-content for students

B. Sc.(honours) Part 2paper 3

Subject:Mathematics

Topic:continuity &differentiability of function of real variable

RRS college mokama

Continuity of function at a point

Definition

The function $f(x)$, defined over the interval I, is said to be continuous at $x = a \in I$ if it possesses a finite limit as x tends to a from either side always remaining in I and each of these limits is equal to $f(a)$.

Thus $f(x)$ is continuous at $x = a$ if

(i) $f(x)$ is defined at $x = a$ and

(ii) $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) = f(a)$

i.e. $$f(a+0) = f(a-0) = f(a)$$

i.e. limit from right = limit from left = value of the function.

If a is an end point of the interval I of definition of $f(x)$ then one of the left-hand limit or right-hand limit exists. In that case, one that exists should be equal to $f(a)$.

Thus if a is a left-end point of the interval I then $f(x)$ is continuous at $x = a$ if $\lim_{x \to a+0} f(x) = f(a)$ and if a is a right-end point of the interval I then $f(x)$ is continuous at $x = a$ if $\lim_{x \to a-0} f(x) = f(a)$.

This definition can be put in the distance form (or modulus form) as follows—

A function $f(x)$, defined over the interval I, is said to be continuous at $x = a \in I$, if given $\epsilon > 0$ there exists a positive number δ such that

$|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in I$

i.e. $f(a) - \epsilon < f(x) < f(a) + \epsilon$ whenever $x \in I$ such that

$$a - \delta < x < a + \delta.$$

A function $f(x)$ which is not continuous at $x = a$ is called *discontinuous* at $x = a$.

Continuity of a function in a interva l :Definition

A function $f(x)$ defined over the interval I is said to be continuous in the interval $[a, b] \subseteq I$ if $f(x)$ is continuous at all points x such that $a \leqslant x \leqslant b$.

Differentiability at a point

A function $f(x)$ defined in the interval I is said to be differentiable at $x = a \in I$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists for all tendencies of x towards a so that x always remains in I.

In other words, the condition is that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

both exist and have the same definite value.

Thus the function $f(x)$ is said to be differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

$$= \text{a definite number.}$$

The value of the limit is called the derivative (differential co-efficient) of $f(x)$ at $x = a$ and it is denoted by

$$f'(a) \text{ or } Df(a).$$

A function $f(x)$ defined in the interval I is said to be differentiable at $x = a \in I$, having the derivative l if given $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - l \right| < \epsilon. \text{ whenever } |x - a| < \delta \text{ and } x \in I.$$

The number l is the derivative of $f(x)$ at $x = a$.

Theorem *A function f is differentiable at $x = a$ if and only if there exists a number l such that*

$$f(a + h) - f(a) = lh + h.\eta$$

where η denotes a quantity which tends to 0 as $h \to 0$.

Proof. Let f be differentiable at $x = a$. Then there exists a number l such that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = l.$

Putting $x = a + h$,

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = l$$

or

$$\lim_{h \to 0} \left[\frac{f(a + h) - f(a)}{h} - l \right] = 0.$$

$$\therefore \quad \frac{f(a + h) - f(a)}{h} - l = \eta \text{ where } \eta \to 0 \text{ as } h \to 0.$$

$$\therefore \quad f(a + h) - f(a) = lh + h.\eta \text{ where } \eta \to 0 \text{ as } h \to 0.$$

Thus it is the necessary condition.

As the argument is reversible, the condition is also sufficient.

Theorem *If a function is differentiable finitely at a point, then it must be continuous at that point.*

Proof. Let the function $f(x)$ be differentiable at $x = a$.

Then by definition $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = A \text{ (say)}.$$

From $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = A$, it follows, by the definition of limit, that, for a given number $\epsilon > 0$, we can find a number $\delta > 0$ such that

$$\left| \frac{f(a+h) - f(a)}{h} - A \right| \leqslant \epsilon, \text{ for } |h| < \delta$$

i.e. $|f(a+h) - f(a) - Ah| \leqslant \epsilon |h|$, for $|h| < \delta$.

But $|f(a+h) - f(a)| - |Ah| \leqslant |f(a+h) - f(a) - Ah|$.

$\therefore$ $|f(a+h) - f(a)| - |Ah| \leqslant \epsilon |h|$.

Hence $|f(a+h) - f(a)| \leqslant |h| (|A| + \epsilon)$.

Now, if $h \to +0$, then $f(a+h) - f(a) \to 0$

and also if $h \to -0$, then $f(a-h) - f(a) \to 0$.

Thus $f(a+0) = f(a) = f(a-0)$ and $f(a)$ is defined.

So $f(x)$ is continuous at $x = a$.

Note. *The converse of this theorem is not necessarily true, i.e. the condition of continuity is not sufficient for differentiability.*

i.e. the continuity of a function is a weaker condition than differentiability. (P U 1967 H)

Let us illustrate this by an example.

Consider the continuity and differentiability of the function $f(x) = |x|$ at $x = 0$.

(M U 1966 H, '78 A'85; Bh U '66 H; AMIE '81)

Test for continuity at $x = 0$.

Limit from right $= f(0+0) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} |0+h| = \lim_{h \to 0} |h| = 0$$

And limit from left $= f(0-0) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} |0-h| = \lim_{h \to 0} |h| = 0.$$

Since $f(0+0) = f(0-0) = f(0) = 0$,

so the given function is continuous at $x = 0$.

Test for differentiability at $x = 0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1, \text{ as } h > 0,$$

and $$\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|0-h| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1.$$

Since these two limits are not equal, therefore the function is not differentiable at $x = 0$.